J. Fluid Mech. (1984), *vol.* 144, *pp.* 311–328
Printed in Great Britain

On the stability of the similarity solutions for swirling flow above an infinite rotating disk

By R. J. BODONYI AND B. S. NG
Department of Mathematical Sciences, Indiana University – Purdue University, Indianapolis, Indiana 46223

(Received 15 November 1983)

A stability theory for the steady swirling flow above an infinite rotating disk immersed in an otherwise unbounded rigidly rotating fluid is developed in order to corroborate the various numerical computations considered for this problem. An analysis of the initial-value problem for linearized time-dependent perturbations on the steady-state similarity solutions shows that the disturbance equations have a stable continuum spectrum which, under certain conditions, exhibits only algebraic decay in time. In addition, a numerical analysis on the discrete spectrum shows that there are unstable eigenvalues for certain rotational rates of the disk relative to the fluid at infinity. The results obtained are in good agreement with the large-time behaviour of the corresponding solutions of the unsteady similarity equations.

1. Introduction

The rotationally symmetric steady flow above an infinite rotating disk immersed in an otherwise unbounded fluid is governed by a set of nonlinear ordinary differential equations. These equations can be derived from the full Navier–Stokes equations using certain similarity transformations, and their integration thus provides a class of exact axisymmetric solutions of the fundamental governing equations of hydrodynamics.

If we define s as the ratio of the angular speed of the rigidly rotating fluid at infinity to that of the disk, then $s = 0$ corresponds to the Kármán swirling flow. It is also convenient to let $\alpha = 1/s$, and $\alpha = 0$ then corresponds to the classical Bödewadt problem. In the case of a rotating disk immersed in a counter-rotating fluid for which $s = \alpha = -1$, McLeod (1970) has proved that a steady-state solution of the similarity equations cannot exist. His conclusion was consistent with the calculations of Evans (1969) and Bodonyi (1975), which suggested that the steady-state solutions can only be obtained outside the interval $-1.435 \lesssim s \lesssim -0.1605$ (or $-6.23 \lesssim \alpha \lesssim -0.6968$). On the other hand, Zandbergen & Dijkstra (1977) and Lentini & Keller (1980) have shown that the steady-state solutions are not unique in a certain bounded neighbourhood of $s = 0$, and that $s \approx -0.1605$ corresponds to the first branch point of an infinite family of multiple solutions. A numerical study by Bodonyi (1978) further indicated that, although solutions of the steady-state similarity equations exist for $\alpha \gtrsim -0.6968$, some of these solutions cannot be obtained as the large-time limit by a time-dependent calculation using the unsteady similarity equations. Instead, for $\alpha = -0.1$ and -0.15, the solutions take on a limit-cycle character for large time; and, for $\alpha < -0.15$, the numerical computations fail to converge following a *blow-up* behaviour similar to that discussed by Bodonyi & Stewartson (1977) and Stewartson, Simpson & Bodonyi (1982) for the case $\alpha = -1$.

In view of these developments, it is desirable to develop a stability theory that will corroborate the various numerical computations. From the physical point of view, any steady-state solution that cannot be obtained as the large-time limit of the corresponding time-dependent Cauchy problem will not be observable. This in turn suggests that the question of physical uniqueness of the Kármán swirling flow can be considered within the framework of the mathematical instability of the recently discovered multiple solutions.

In this paper, a systematic study is made on the stability of the steady similarity solutions. By considering the initial-value problem for linearized time-dependent perturbations on the steady-state solutions, we show that the disturbance equations have a continuous spectrum containing only stable modes. However, a numerical analysis of the discrete spectrum shows that it may have unstable eigenvalues for certain values of α or s. In particular, we show that the recently discovered multiple solutions in the neighbourhood of $s = 0$, as well as those solutions in the range $-0.6968 \lesssim \alpha \lesssim -0.03$, are indeed linearly unstable in a Cauchy sense. Our results are thus in good agreement with the large-time behaviour of the corresponding time-dependent solutions of the unsteady similarity equations.

2. Problem formulation

The governing equations for the basic swirling flow above an infinite rotating disk can readily be obtained from the axisymmetric Navier–Stokes equations written in the usual polar coordinates $(\hat{r}, \theta, \hat{z})$. Here we let $\hat{z} = 0$ be the plane of rotation of the disk and Ω_0 and Ω_∞ be respectively the angular speed of the disk and that of the rigidly rotating fluid at infinity. We also introduce a reference angular speed Ω, which can be chosen as either Ω_0 or Ω_∞. For our purposes, however, it is not necessary to fix Ω until later. If we now let $\hat{u}$, $\hat{v}$ and $\hat{w}$ be the velocity components in the $\hat{r}$-, θ- and $\hat{z}$-directions respectively, and let $\hat{p}$ be the pressure, then, following Kármán (1921), we seek unsteady similarity solutions of the form

$$[\hat{u}, \hat{v}, \hat{w}] = (\nu\Omega)^{\frac{1}{2}}[rf_z(z,t), rg(z,t), -2f(z,t)], \tag{2.1a}$$

$$\hat{p} = \rho\nu\Omega[\tfrac{1}{2}\gamma^2r^2 + h(z,t)], \tag{2.1b}$$

where $\gamma = \Omega_\infty/\Omega$, ρ is the density, ν is the kinematic viscosity, and

$$r = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\hat{r}, \quad z = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\hat{z}, \quad t = \Omega\hat{t}. \tag{2.2}$$

On substituting (2.1) and (2.2) into the Navier–Stokes equations, we obtain

$$f_{zt} = f_{zzz} + 2ff_{zz} - f_z^2 + g^2 - \gamma^2, \tag{2.3a}$$

$$g_t = g_{zz} + 2fg_z - 2gf_z, \tag{2.3b}$$

$$h_z = 2(f_t - 2ff_z - f_{zz}), \tag{2.3c}$$

where subscripts denote partial derivatives. We note that in (2.1) the term $\frac{1}{2}\rho\nu\Omega\gamma^2r^2$ in $\hat{p}$ corresponds to the inviscid pressure, and that the equation for $h(z,t)$ is uncoupled from the equations for $f(z,t)$ and $g(z,t)$ in (2.3). It will be sufficient, therefore, to consider only (2.3a,b) in our subsequent analysis. Moreover, apart from the equation for $h(z,t)$, (2.3) is identical with the similarity equations obtained by substituting (2.1) and (2.2) into the boundary-layer equations. In this instance, however, the terms neglected in the boundary-layer approximation are identically zero under the similarity transformation.

If we now let $\Omega_0 \neq 0$ be the reference angular speed and $s = \Omega_\infty/\Omega_0$, then $\gamma^2 = s^2$. The no-slip condition at the disk and the rigid rotation of the fluid at infinity then imply for all $t > 0$

$$f = f_z = 0, \quad g = 1 \quad \text{at} \quad z = 0, \tag{2.4a}$$

and

$$f_z \rightarrow 0, \quad g \rightarrow s \quad \text{as} \quad z \rightarrow \infty. \tag{2.4b}$$

On the other hand, if we let $\Omega = \Omega_\infty \neq 0$ and $\alpha = \Omega_0/\Omega_\infty$, then $\gamma^2 = 1$, and the boundary conditions for all $t > 0$ become

$$f = f_z = 0, \quad g = \alpha \quad \text{at} \quad z = 0, \tag{2.5a}$$

and

$$f_z \rightarrow 0, \quad g \rightarrow 1 \quad \text{as} \quad z \rightarrow \infty. \tag{2.5b}$$

Equations (2.3) together with (2.4) or (2.5) and some appropriate initial conditions at $t = 0$ define a Cauchy initial–boundary-value problem, the solutions of which are also solutions of the full Navier–Stokes equations. To study the stability of these solutions in the limit $t \rightarrow \infty$, we consider the effects of small perturbations on the corresponding steady-state solutions. If a solution of (2.3) is stable, then all small perturbations of the corresponding steady-state solution should ultimately decay to zero as $t \rightarrow \infty$, leaving the basic steady flow unchanged. Conversely, the flow is said to be unstable to infinitesimal disturbances if any of the disturbances does not decay to zero, or if the flow evolves into a new steady state.

To derive the appropriate disturbance equations, we write

$$f(z,t) = f_0(z) + \tilde{f}(z,t) \quad g(z,t) = g_0(z) + \tilde{g}(z,t), \tag{2.6a, b}$$

where f_0 and g_0 satisfy the steady-state form of (2.3), i.e. $(\ )_t = 0$, and $\tilde{f}$ and $\tilde{g}$ are small time-dependent perturbations. On substituting (2.6) into (2.3a, b) and linearizing, we obtain

$$\tilde{f}_{zt} = \tilde{f}_{zzz} + 2(f_0 \tilde{f}_{zz} - f_0' \tilde{f}_z + f_0'' \tilde{f} + g_0 \tilde{g}), \tag{2.7a}$$

$$\tilde{g}_t = \tilde{g}_{zz} + 2(f_0 \tilde{g}_z - f_0' \tilde{g} - g_0 \tilde{f}_z + g_0' \tilde{f}). \tag{2.7b}$$

It follows from (2.6) and either (2.4) or (2.5) that the boundary conditions for all t are given by

$$\tilde{f} = \tilde{f}_z = \tilde{g} = 0 \quad \text{at} \quad z = 0, \tag{2.8a}$$

and

$$\tilde{f}_z \rightarrow 0, \quad \tilde{g} \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty. \tag{2.8b}$$

Equations (2.7) and (2.8) then provide the starting point of our analysis. In §3 a study is made of the behaviour of the continuum spectrum of (2.7)–(2.8). A normal-mode analysis of the discrete spectrum of these equations is given in §4.

3. The initial-value problem and the continuum spectrum

To solve (2.7) and (2.8) with some appropriate initial conditions, we apply the Laplace transform in t such that

$$\bar{f}(z,\lambda) = \int_0^\infty e^{-\lambda t} \tilde{f}(z,t)\, dt, \tag{3.1}$$

and similarly for $\tilde{g}(z,t)$. Then, on omitting $(\bar{\ })$ from the various transformed quantities, (2.7) becomes

$$\left.\begin{aligned} f''' + 2(f_0 f'' - f_0' f' + f_0'' f + g_0 g) - \lambda f' &= -\tilde{f}'(z,0), \\ g'' + 2(f_0 g' - f_0' g - g_0 f' + g_0' f) - \lambda g &= -\tilde{g}(z,0), \end{aligned}\right\} \tag{3.2}$$

where $(\ )' = \mathrm{d}/\mathrm{d}z$, and $\tilde{f}(z,0)$ and $\tilde{g}(z,0)$ are the initial values of $\tilde{f}$ and $\tilde{g}$ at $t=0$. If we let $\boldsymbol{\phi} = [f, f', f'', g, g']^{\mathrm{T}}$, (3.2) can be written as a first-order inhomogeneous system of the form

$$\boldsymbol{\phi}' = \boldsymbol{A}\boldsymbol{\phi} - \boldsymbol{h}, \tag{3.3}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2f_0'' & \lambda+2f_0' & -2f_0 & -2g_0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2g_0' & 2g_0 & 0 & \lambda+2f_0' & -2f_0 \end{bmatrix}, \tag{3.4}$$

$$\boldsymbol{h} = [0, 0, h_3, 0, h_5]^{\mathrm{T}}, \tag{3.5}$$

with

$$h_3 = \tilde{f}'(z,0), \quad h_5 = \tilde{g}(z,0). \tag{3.6}$$

The boundary conditions for (3.3) are identical with those given in (2.8) except that $\tilde{f}$, $\tilde{f}_z$, and $\tilde{g}$ in (2.8) must be replaced by f, f' and g respectively.

Consider now the solution of (3.3) by the method of complementary functions. First we note that, as $z \to \infty$, $f_0 = f_\infty$, $f_0' = f_0'' = 0$, $g_0 = \gamma$ and $g_0' = 0$, where f_∞ is a constant which depends on α (or s). Thus the coefficient matrix becomes

$$\boldsymbol{A}_\infty = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda & -2f_\infty & -2\gamma & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2\gamma & 0 & \lambda & -2f_\infty \end{bmatrix}. \tag{3.7}$$

The eigenvalues of $\boldsymbol{A}_\infty$ are given by

$$\beta_1 = 0, \quad \beta_{2,4} = -f_\infty \mp \mu_+, \quad \beta_{3,5} = -f_\infty \mp \mu_-, \tag{3.8}$$

where $\mu_\pm = (f_\infty^2 + \lambda \pm 2\gamma\mathrm{i})^{\frac{1}{2}}$ and $\mathrm{Re}\,(\mu_\pm) > 0$. We thus define five linearly independent homogeneous solutions of (3.3), i.e.

$$\boldsymbol{\phi}_1 = \xi_1 \quad \text{and} \quad \boldsymbol{\phi}_k \sim \xi_k \exp(\beta_k z) \quad (k = 2, \ldots, 5) \quad \text{as} \quad z \to \infty, \tag{3.9}$$

where

$$\left.\begin{aligned} \xi_1 &= [1, 0, 0, 0, 0]^{\mathrm{T}}, \\ \xi_2 &= [1, \beta_2, \beta_2^2, -\mathrm{i}\beta_2, -\mathrm{i}\beta_2^2]^{\mathrm{T}}, \\ \xi_3 &= [1, \beta_3, \beta_3^2, \mathrm{i}\beta_3, \mathrm{i}\beta_3^2]^{\mathrm{T}}, \\ \xi_4 &= [1, \beta_4, \beta_4^2, -\mathrm{i}\beta_4, -\mathrm{i}\beta_4^2]^{\mathrm{T}}, \\ \xi_5 &= [1, \beta_5, \beta_5^2, \mathrm{i}\beta_5, \mathrm{i}\beta_5^2]^{\mathrm{T}}. \end{aligned}\right\} \tag{3.10}$$

Before proceeding, it is useful to consider the behaviour of these solutions in the complex λ-plane which is summarized in figure 1 for $\gamma > 0$. First we note that $\boldsymbol{\phi}_1$ is bounded and it automatically satisfies the boundary conditions (2.8*b*) at infinity for all values of λ and f_∞. On the other hand, the behaviour of $\boldsymbol{\phi}_k$ $(k = 2, \ldots, 5)$ depends crucially on f_∞.

In the case of $f_\infty \geqslant 0$, $\mathrm{Re}\,(\beta_2)$ and $\mathrm{Re}\,(\beta_3)$ are always < 0, and $\boldsymbol{\phi}_2$ and $\boldsymbol{\phi}_3$ are therefore always bounded as $z \to \infty$. The solutions $\boldsymbol{\phi}_4$ and $\boldsymbol{\phi}_5$ are, however, bounded only if $\lambda_{\mathrm{r}} < p_+$ and $\lambda_{\mathrm{r}} < p_-$ respectively, where $p_\pm = -(\lambda_{\mathrm{i}} \pm 2\gamma)^2/4f_\infty^2$. We note that the two branch cuts shown in figure 1 are placed so as to render $\mathrm{Re}\,(\mu_\pm) > 0$. Moreover, when $f_\infty = 0$ the parabolas defined by $\lambda_{\mathrm{r}} = p_\pm$ degenerate, and they coincide with the branch cuts emanating from $\lambda = \pm 2\gamma\mathrm{i}$.

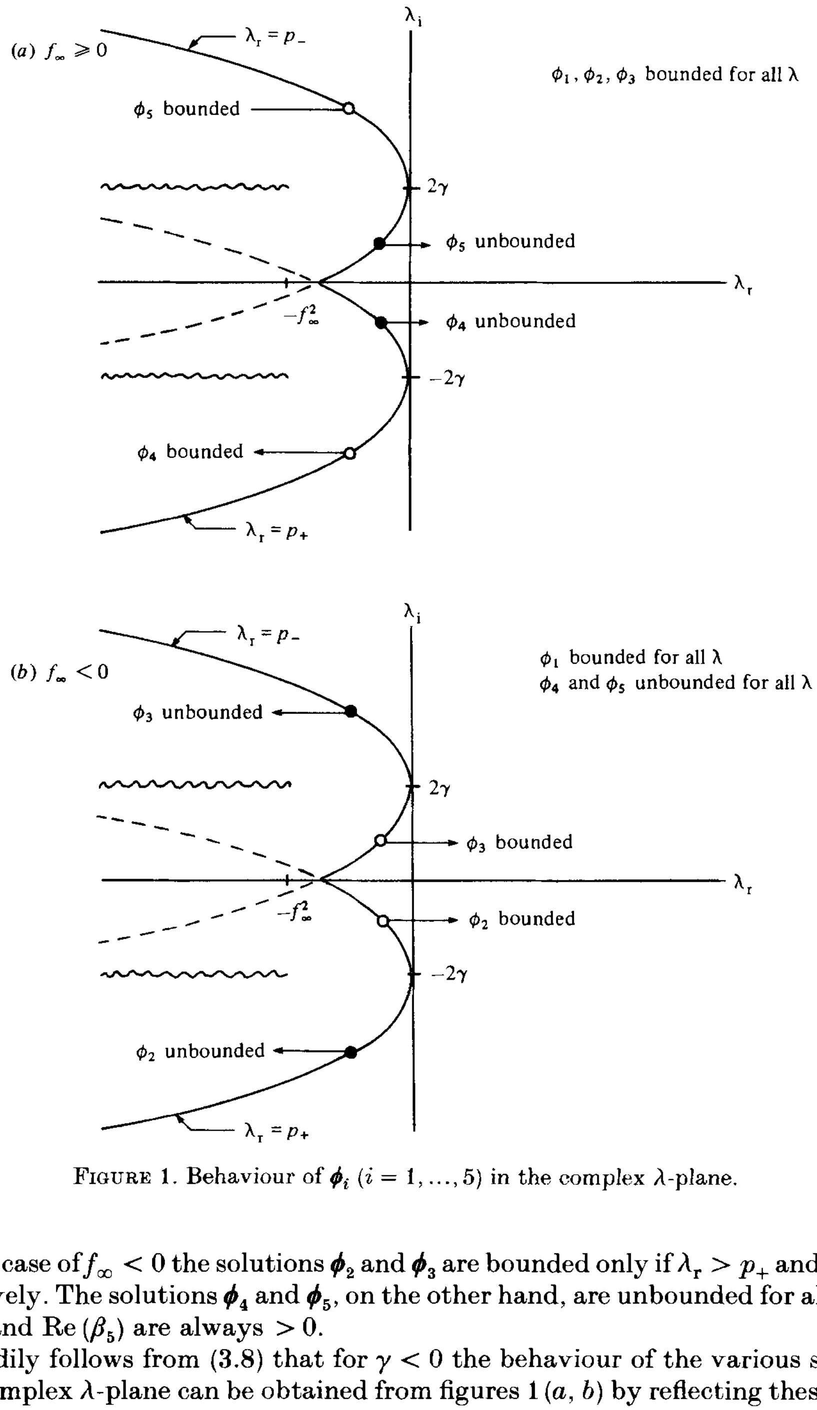

FIGURE 1. Behaviour of $\boldsymbol{\phi}_i$ $(i = 1, \ldots, 5)$ in the complex λ-plane.

In the case of $f_\infty < 0$ the solutions $\boldsymbol{\phi}_2$ and $\boldsymbol{\phi}_3$ are bounded only if $\lambda_r > p_+$ and $\lambda_r > p_-$ respectively. The solutions $\boldsymbol{\phi}_4$ and $\boldsymbol{\phi}_5$, on the other hand, are unbounded for all λ since $\mathrm{Re}\,(\beta_4)$ and $\mathrm{Re}\,(\beta_5)$ are always > 0.

It readily follows from (3.8) that for $\gamma < 0$ the behaviour of the various solutions in the complex λ-plane can be obtained from figures 1 (a, b) by reflecting these figures about their respective λ_r axis.

A simple calculation shows that the Wronskian of the 5×5 homogeneous solution matrix $\boldsymbol{\Phi} = [\boldsymbol{\phi}_k]$ is given by $W(z) = |\boldsymbol{\Phi}(z)| = W_\infty / w(z)$, where

$$W_\infty = 16(\lambda^2 + 4\gamma^2)\,\mu_+\,\mu_-\exp\left(-4f_\infty z_\infty\right), \quad w(z) = \exp\left(4\int_{z_\infty}^{z} f_0\,\mathrm{d}z\right), \qquad (3.11)$$

and z_∞ must be chosen such that for $z \geqslant z_\infty$ the coefficient matrix $\boldsymbol{A}$ is sensibly constant. Using the method of variation of parameters, the solution of (3.3) can then be written as

$$\boldsymbol{\phi} = \sum_{k=1}^{5} A_k(z)\,\boldsymbol{\phi}_k(z), \tag{3.12}$$

where

$$A_k(z) = \frac{1}{W_\infty}\int_{z_k}^{z} H_k(z)\,w(z)\,\mathrm{d}z. \tag{3.13}$$

Here $H_k(z) = -(C_{3k}h_3 + C_{5k}h_5)$, with C_{jk} being the cofactor of ϕ_{kj} in $\boldsymbol{\Phi}$. The lower limits of integration z_k must be chosen such that the boundary conditions (2.8) are satisfied. It readily follows from (2.8b) and the behaviour of $\boldsymbol{\phi}_4$ and $\boldsymbol{\phi}_5$ at infinity that $z_4 = z_5 = \infty$. For convenience we write

$$\int_{z_k}^{z} H_k(z)\,w(z)\,\mathrm{d}z = a_k + \int_0^z H_k(z)\,w(z)\,\mathrm{d}z \quad (k = 1,2,3) \tag{3.14}$$

such that

$$\boldsymbol{\phi} = \frac{1}{W_\infty}\bigg[\boldsymbol{\phi}_1\Big(a_1 + \int_0^z H_1 w\,\mathrm{d}z\Big) + \boldsymbol{\phi}_2\Big(a_2 + \int_0^z H_2 w\,\mathrm{d}z\Big) + \boldsymbol{\phi}_3\Big(a_3 + \int_0^z H_3 w\,\mathrm{d}z\Big)$$
$$+ \boldsymbol{\phi}_4\int_\infty^z H_4 w\,\mathrm{d}z + \boldsymbol{\phi}_5\int_\infty^z H_5 w\,\mathrm{d}z\bigg]. \tag{3.15}$$

The boundary conditions at $z = 0$ then imply

$$\left.\begin{aligned} a_1 &= -(a_4E_{234} + a_5E_{235})/E_{123},\\ a_2 &= (a_4E_{134} + a_5E_{135})/E_{123},\\ a_3 &= -(a_4E_{124} + a_5E_{125})/E_{123},\end{aligned}\right\} \tag{3.16}$$

where

$$a_4 = \int_\infty^0 H_4 w\,\mathrm{d}z, \quad a_5 = \int_\infty^0 H_5 w\,\mathrm{d}z \tag{3.17}$$

and

$$E_{ijk} = \det\begin{bmatrix} f_i(0) & f_j(0) & f_k(0)\\ f_i'(0) & f_j'(0) & f_k'(0)\\ g_i(0) & g_j(0) & g_k(0)\end{bmatrix}. \tag{3.18}$$

The solution of (2.7) and (2.8) can now be obtained from (3.15) by the inverse Laplace transform. If we let $\tilde{\boldsymbol{\phi}} = [\tilde{f}, \tilde{f}', \tilde{f}'', \tilde{g}, \tilde{g}']^{\mathrm{T}}$ then

$$\tilde{\boldsymbol{\phi}} = \frac{1}{2\pi\mathrm{i}}\int_{p-\mathrm{i}\infty}^{p+\mathrm{i}\infty} \mathrm{e}^{\lambda t}\boldsymbol{\phi}\,\mathrm{d}\lambda, \tag{3.19}$$

where the constant p must be sufficiently positive so that all singularities of $\boldsymbol{\phi}$ lie to the left of the line $\mathscr{C}_0$ (say) from $p - \mathrm{i}\infty$ to $p + \mathrm{i}\infty$. In order to evaluate the integral (3.19) by the method of residues, the path $\mathscr{C}_0$ must be closed to the left so that all the poles of $\boldsymbol{\phi}$ are enclosed. The appropriate enclosing paths of integration for $f_\infty \geqslant 0$ and $f_\infty < 0$ are shown in figures 2(a) and (b) respectively. In particular, we note that for $f_\infty < 0$ the choice of that portion of the closed contour along the parabolic curves $\lambda_r = p_\pm$ is dictated by the fact that $\boldsymbol{\phi}$ must be kept bounded as $z \to \infty$.

It is easily shown that the contribution of the integral along $\mathscr{C}_R$ vanishes as $R \to \infty$.

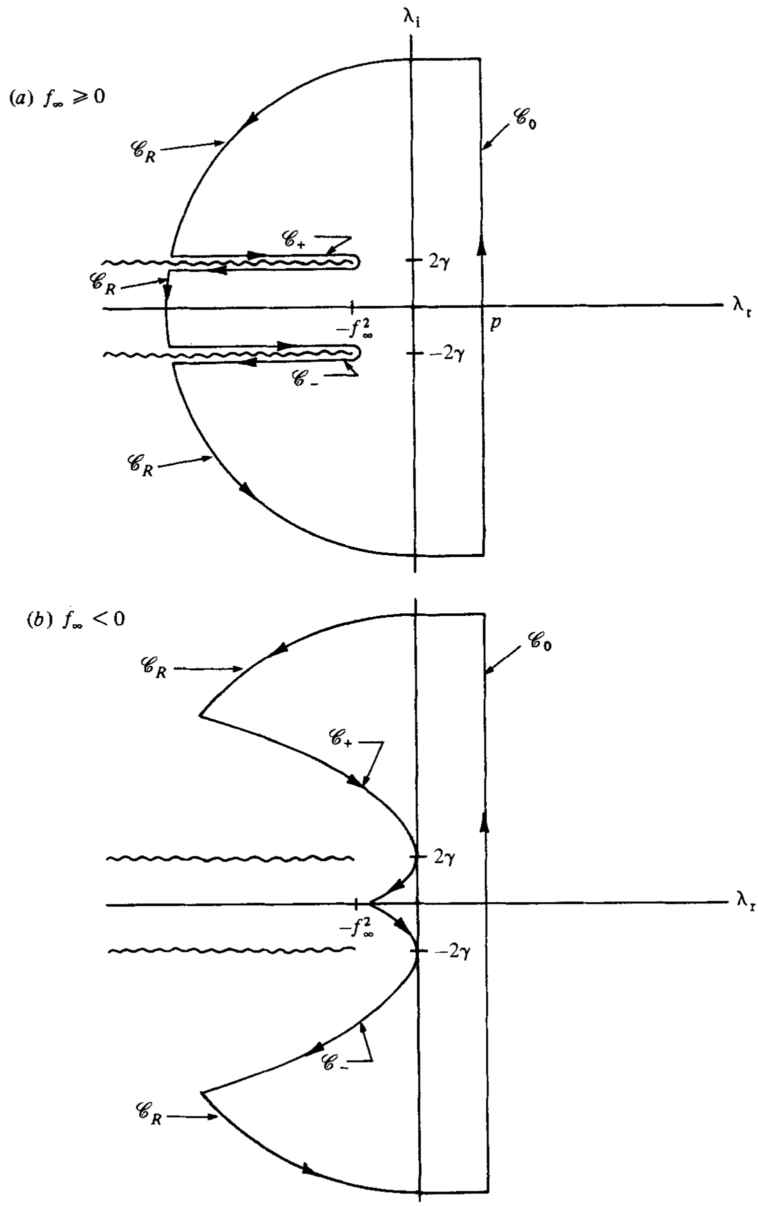

FIGURE 2. Paths of integration for the inverse Laplace transform.

Thus (3.19) becomes

$$\tilde{\boldsymbol{\phi}} = \sum_k \mathrm{Res}\,[\exp(\lambda_k t)\,\boldsymbol{\phi}(\lambda_k)] - \frac{1}{2\pi \mathrm{i}}\left[\int_{\mathscr{C}_+} + \int_{\mathscr{C}_-} \exp(\lambda t)\,\boldsymbol{\phi}\,\mathrm{d}\lambda\right], \tag{3.20}$$

where Res denotes the residue of $\exp(\lambda t)\,\boldsymbol{\phi}$ at the pole λ_k. Using an argument similar to that of Gustavsson (1979) in his study of the Orr–Sommerfeld problem, one can deduce from (3.15)–(3.18) that the poles of $\boldsymbol{\phi}$ are determined by $E_{123} = 0$, which is the ordinary eigenvalue relation for the normal modes of (2.7) and (2.8) (see §4). Moreover, the sum of the integrals along $\mathscr{C}_+$ and $\mathscr{C}_-$ represents the contribution to

$\tilde{\phi}$ from the continuum spectrum of (2.7) and (2.8). We also recall that for $f_\infty \geqslant 0$ and $\lambda_r < p_+$ or $\lambda_r < p_-$ there exist no fewer than four decaying homogeneous solutions of (3.3). Thus the values of λ lying to the left of the parabolic curves $\lambda_r = p_\pm$ appear to form a continuum of eigenvalues for which non-trivial homogeneous solutions of (3.3) can always be found which satisfy the three homogeneous boundary conditions at $z = 0$. Nevertheless, (3.20) shows that these continuous modes are not needed for the spectral resolution of the present initial-value problem.

For general values of f_∞ the integrals in (3.20) are of very complicated form, and a complete analysis of the properties of the continuum spectrum based on (3.20) does not appear to be possible at the present time. However, by a somewhat tedious calculation (cf. Gustavsson 1979, pp. 1604–1605), we can show that the initial disturbance represented by the continuum spectrum will decay as $t \to \infty$ for all values of f_∞. In the case of $f_\infty > 0$ the decay rate is exponential and its time dependence is given by the factor $t^{-\frac{3}{2}}\exp(-f_\infty^2 t)$. For $f_\infty \leqslant 0$ the decay rate is algebraic and it is dominated by a leading term proportional to $t^{-\frac{3}{2}}$.

Alternatively, these results can perhaps be seen more clearly by considering the large-time behaviour of the continuum modes as $z \to \infty$ using an approach similar to that used by Murdock & Stewartson (1977) in their study of the spectra of the Orr–Sommerfeld equation. To this end, we multiply (2.7b) by i and add to (2.7a) to obtain

$$(\tilde{f}_z + \mathrm{i}\tilde{g})_t = (\tilde{f}_z + \mathrm{i}\tilde{g})_{zz} + 2(f_0 - f_0')(\tilde{f}_z + \mathrm{i}\tilde{g})_z - 2\mathrm{i}g_0(\tilde{f}_z + \mathrm{i}\tilde{g}) + 2(f_0'' + \mathrm{i}g_0')\tilde{f}. \tag{3.21}$$

If we let

$$\chi = \tilde{f}_z + \mathrm{i}\tilde{g}, \tag{3.22}$$

then in the limit $z \to \infty$ (3.21) becomes

$$\chi_t = \chi_{zz} + 2f_\infty \chi_z - 2\gamma \mathrm{i}\chi. \tag{3.23}$$

Note that (3.23) is valid for *all* values of α for z sufficiently large, and it is of the heat-conduction type. Thus if we let

$$\chi = \exp[-f_\infty z - f_\infty^2 t - 2\gamma \mathrm{i}t]\,\psi(z,t), \tag{3.24}$$

it is easily shown that (3.23) reduces to the heat equation in ψ, i.e.

$$\psi_t = \psi_{zz}. \tag{3.25}$$

To study the *large-time* behaviour of (3.25), it is useful to introduce a further change of variables by writing

$$\eta = \frac{z}{(2t)^{\frac{1}{2}}}, \quad \psi(z,t) = t^{-\frac{1}{2}q}\,\mathrm{e}^{-\frac{1}{4}\eta^2}\psi_q(\eta), \tag{3.26}$$

where q is an arbitrary constant and $\psi_q(\eta)$ must satisfy

$$\psi_q'' + [q - \tfrac{1}{2} - \tfrac{1}{4}\eta^2]\,\psi_q = 0. \tag{3.27}$$

Solutions of (3.27) can be expressed in terms of the Hermite polynomials $H_{q-1}(\eta)$ when q is an integer.

In the case of solid-body rotation for which $f_\infty = 0$ and $\gamma = 1$, (3.23) is exact for all z and the boundary condition at $\eta = 0$ requires that $\psi_q(0) = 0$. This in turn implies that

$$q = 2(n+1) \quad (n = 0, 1, 2, \ldots). \tag{3.28}$$

Thus the solution of (3.23) is given by

$$\chi_n = c_n\, t^{-(n+1)}\, \mathrm{e}^{-(2\mathrm{i}t+z^2/4t)}\, H_{2n+1}\left(\frac{z}{2t^{\frac{1}{2}}}\right), \tag{3.29}$$

where the constant c_n is dependent on the initial condition on χ. However, since our interest here is the ultimate growth or decay of χ as $t \rightarrow \infty$, the precise form of the initial condition need not concern us. It is sufficient for our purposes to note that for $t \rightarrow \infty$ with z fixed the leading term of (3.29) gives

$$\chi \propto z t^{-\frac{3}{2}}\, \mathrm{e}^{-2\mathrm{i}t}, \tag{3.30}$$

which shows that the disturbance does approach zero as $t \rightarrow \infty$ although the decay is only algebraic.

In the general case for which (3.23) is valid only in the limit $z \rightarrow \infty$, the arbitrary constant q in the solution of (3.23) can, in principle, be determined by matching to the solution of (3.21) which is valid for z of order one. On the other hand, we can also argue that the form of the solution of (3.23) for arbitrary α must be such that it be reducible to (3.29) in the limit of solid-body rotation. This then fixes q as in (3.28). Thus, in general, we have

$$\chi_n = c_n t^{-(n+1)} \exp\left[\left(\frac{z+2f_\infty t}{2t^{\frac{1}{2}}}\right)^2 - 2\gamma \mathrm{i}t\right] H_{2n+1}\left(\frac{z}{2t^{\frac{1}{2}}}\right), \tag{3.31}$$

which is valid as $z \rightarrow \infty$. It readily follows from (3.31) that, when $f_\infty > 0$, χ and therefore $\tilde{f}$ and $\tilde{g}$ decay exponentially everywhere in the free stream as $t \rightarrow \infty$. However, for $f_\infty < 0$, $|\chi|$ has a maximum value $\sim t^{-\frac{3}{2}}$ occurring at $z \approx -2f_\infty t$. Thus when $f_\infty \leqslant 0$ it is possible to have an algebraic decay in t as $t \rightarrow \infty$.

4. The discrete spectrum

Within the framework of a normal-mode analysis, we seek solutions of (2.7) of the form

$$\tilde{f}(z,t) = \mathrm{e}^{\lambda t}F(z), \quad \tilde{g}(z,t) = \mathrm{e}^{\lambda t}G(z). \tag{4.1}$$

Equations (2.7) then become

$$\boldsymbol{\phi}' = \boldsymbol{A}\boldsymbol{\phi}, \tag{4.2}$$

where we now have $\boldsymbol{\phi} = [F, F', F'', G, G']^{\mathrm{T}}$, and $\boldsymbol{A}$ is defined by (3.4). The homogeneous boundary conditions are

$$F = F' = G = 0 \quad \text{at} \quad z = 0, \tag{4.3a}$$

and

$$F' \rightarrow 0, \quad G \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty. \tag{4.3b}$$

Equations (4.2) and (4.3) define a fifth-order singular eigenvalue problem on the semi-infinite interval $[0, \infty)$. Moreover, using the definition (3.9) and (3.10), we let $\boldsymbol{\phi}_k$ $(k = 1, 2, 3)$ be the three bounded solutions of (4.2). Then, in the notation of §3 (cf. (3.18)), the eigenvalue relation for the discrete modes is given by

$$E_{123} = 0. \tag{4.4}$$

A description of the compound matrix method for the determination of the eigenvalue λ based on (4.4) is given in the Appendix.

α	λ_{1r} or λ_{2r}	λ_{1i} or $-\lambda_{2i}$
0.038	−0.03839	1.74565
0.036	−0.03746	1.74366
0.035	−0.03699	1.74266
0.030	−0.03465	1.73764
0.020	−0.02986	1.72747
0.000	−0.01983	1.70659
−0.020	−0.009196	1.68498
−0.030	−0.003653	1.67389
−0.040	0.002044	1.66261
−0.060	0.01391	1.63947
−0.080	0.02643	1.61553
−0.100	0.03963	1.59075
−0.200	0.11687	1.45283
−0.300	0.21787	1.28483
−0.400	0.35597	1.06574
−0.500	0.56553	0.71114
−0.550	0.72875	0.28014
−0.555	0.74898	0.16475
	λ_1	λ_2
−0.560	0.93277	0.60755
−0.580	1.38524	0.34754
−0.600	1.74845	0.22693
−0.620	2.15300	0.14532
−0.640	2.68043	0.08573
−0.665	3.82216	0.03330
−0.675	4.71855	0.01838

TABLE 1. The eigenvalues λ_1 and λ_2 for $-0.675 \leqslant \alpha \leqslant 0.038$

4.1. *The discrete modes for* $-0.675 \leqslant \alpha \leqslant 0.038$

First we consider the steady similarity solutions in the neighbourhood of the Bödewadt problem. Thus we let Ω_∞ be the reference angular speed of the basic flow such that $\alpha = \Omega_0/\Omega_\infty$, and we set $\gamma^2 = 1$, $f_{zt} = g_t = 0$ in (2.3). The resulting steady-state equations, together with the boundary conditions (2.5), are solved for a selected range of α by a shooting technique similar to the one used by Zandbergen & Dijkstra (1977). These solutions are then used in conjunction with (4.2)–(4.4) for the stability calculations.

We begin with the case $\alpha = 0$, which corresponds to the classical Bödewadt solution. Once we have obtained an eigenvalue for $\alpha = 0$, the stability calculations can be extended to other values of α by incrementing α in small steps. At each stage, the value of λ previously obtained at a nearby value of α can be used as the initial guess to start the iterative search for the eigenvalue at the current value of α. The eigenvalues λ_1 and λ_2 thus obtained for $-0.675 \leqslant \alpha \leqslant 0.038$ are tabulated in table 1 and they are also shown in figure 3. Note that for $\alpha \lesssim -0.557$, λ_1 and λ_2 are real, but they coalesce at $\alpha \approx -0.557$ and then form a complex-conjugate pair with $\lambda_2 = \lambda_1^*$ for $\alpha \gtrsim -0.557$. The overall trend in figure 3 also seems to suggest that $\lambda_1 \rightarrow \infty$ and $\lambda_2 \rightarrow 0$ as $\alpha \rightarrow -0.6968$, at which point the steady-state equations become singular.

Of special interest is the value of λ_{1r}, since it governs the growth or decay of the perturbation functions $\tilde{f}(z,t)$ and $\tilde{g}(z,t)$. As can be seen from figure 3, λ_{1r} is negative for $-0.03 \lesssim \alpha \lesssim 0.038$, indicating that the perturbations decay to zero as $t \rightarrow \infty$. We

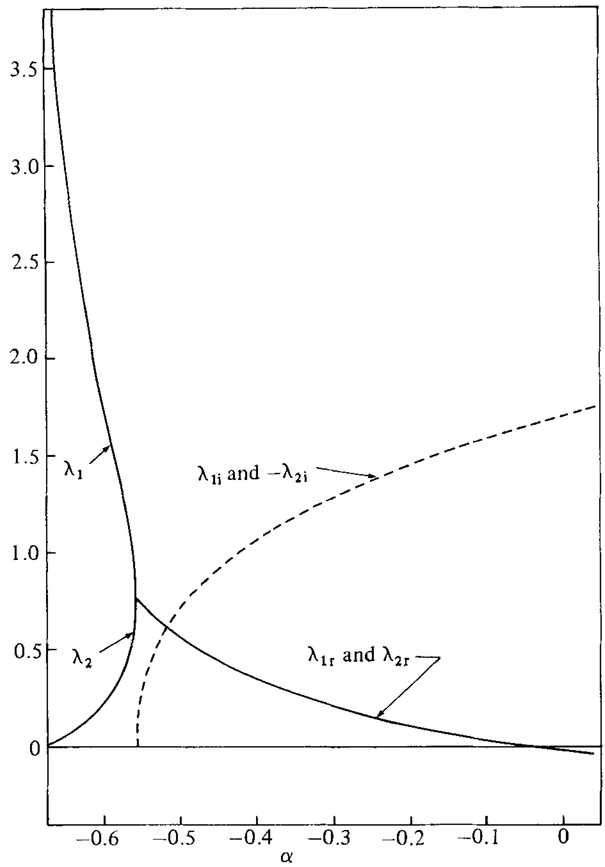

FIGURE 3. The eigenvalues λ_1 and λ_2 for $-0.675 \leqslant \alpha \leqslant 0.038$.

note that the locus of λ_1 does not extend into the region $\alpha \gtrsim 0.04$. In fact, a closer analysis shows that the locus of λ_1 must terminate at $\alpha \approx 0.04$, where it intersects with that part of the continuum spectrum defined by $\lambda_{1r} = p_-$ (see figure 1). It is also of some interest to consider the behaviour of the eigenfunction $\boldsymbol{\phi}$ near $\alpha = 0.04$. An examination of the values of β_3 near $\alpha = 0.04$ shows that, while $\mathrm{Re}\,(\beta_3)$ is negative for $\alpha \lesssim 0.04$, it increases toward 0 as α approaches 0.04 along the locus of λ_1. In figures 4(*a*) and (*b*) we present the F' component of the eigenfunction at $\alpha = 0$ and $\alpha = 0.03$, respectively. A comparison between these figures clearly shows a decrease in the decay rate of F' (as $z \rightarrow \infty$) at $\alpha = 0.03$ as a consequence of the increase of $\mathrm{Re}\,(\beta_3)$ toward 0. The highly oscillatory behaviour of the eigenfunction of a discrete mode in the neighbourhood of the continuous spectrum is similar to that exhibited by certain continuum modes of the Orr–Sommerfeld problem in a semi-infinite domain (cf. Grosch & Salwen 1978). Although the present search for eigenvalues cannot be claimed to be exhaustive, our results suggest that in the interval $0.04 \lesssim \alpha \lesssim 1$ (say), only continuum modes exist which are stable. The steady-state solutions are therefore stable to small perturbations for $-0.03 \lesssim \alpha \lesssim 1$. On the other hand, when $\alpha \lesssim -0.03$, λ_{1r} is positive, implying that the steady-state solutions are ultimately unstable to infinitesimal disturbances. These results are in agreement with the large-time behaviour of the unsteady nonlinear similarity solutions discussed by Bodonyi (1978).

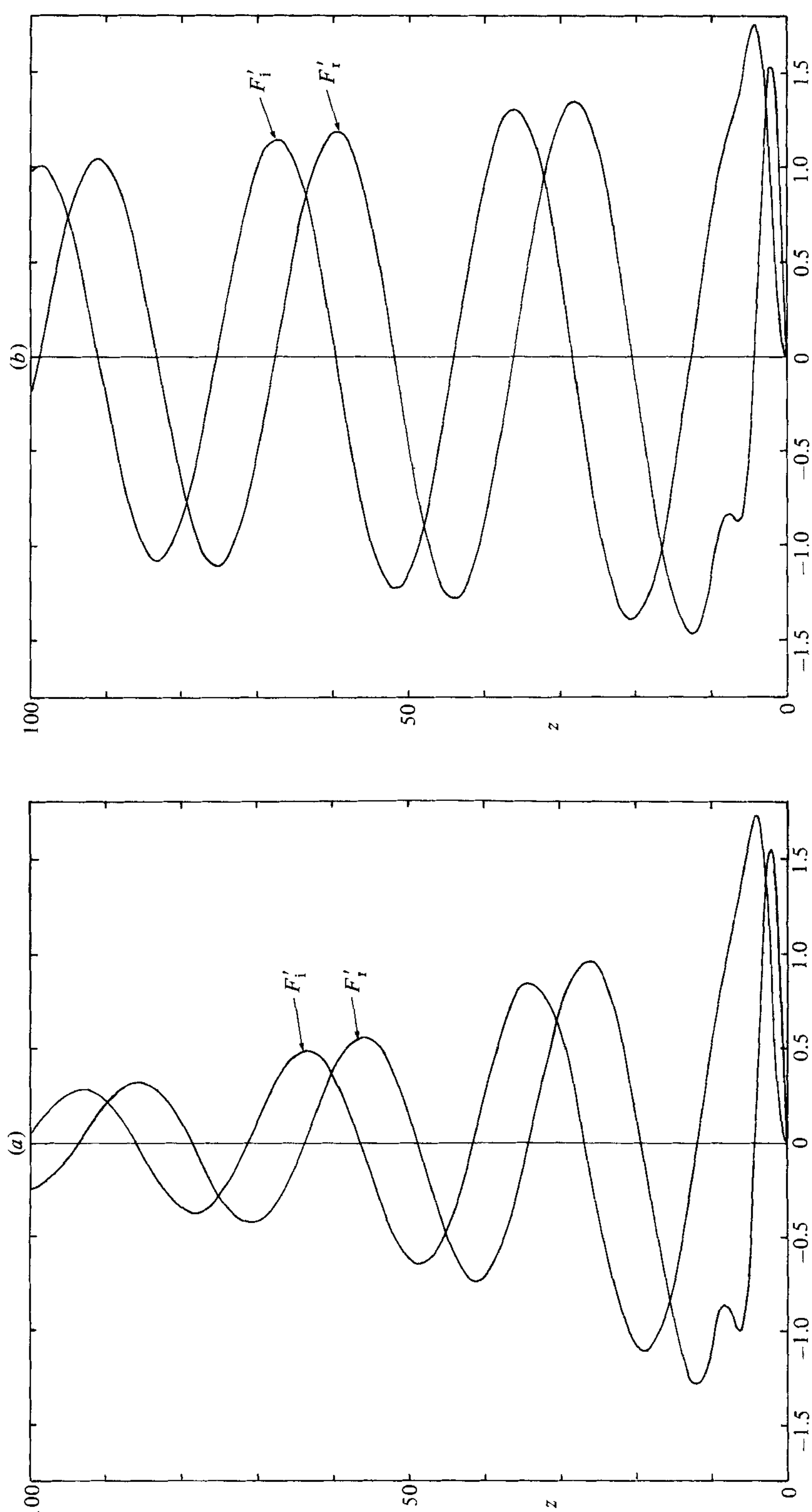

FIGURE 4. The radial eigenfunction $F' = F'_r + iF'_i$: (*a*) for the Bödewadt solution $\alpha = 0$; (*b*) for $\alpha = 0.03$.

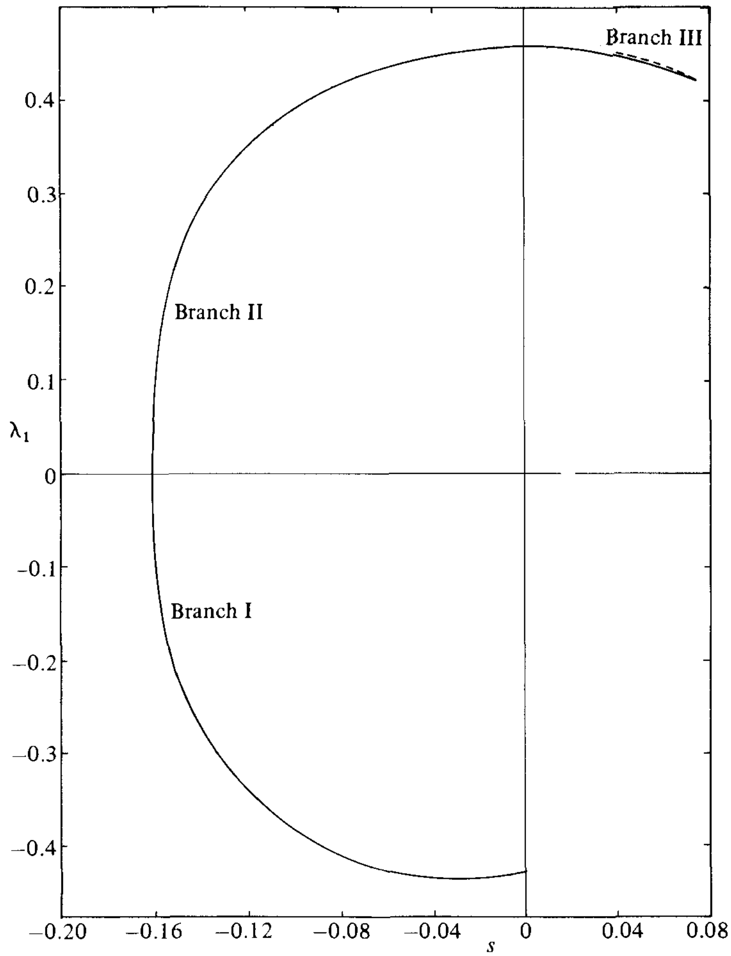

FIGURE 5. The eigenvalue λ_1 as a function of s.

We also note that the eigenvalue problem (4.2)–(4.4) for $-0.55 \leqslant \alpha \leqslant 1$ has been studied earlier by Bodonyi (1973) using a finite-difference scheme and with the boundary conditions at infinity imposed at $z = 30.3$ and 60.6. Because of the replacement of the semi-infinite interval by a truncated interval, however, it appears that many of the eigenvalues obtained were spurious. In addition to λ_1 and λ_2, the finite-difference calculations of Bodonyi have also produced a number of higher modes for $-0.55 \leqslant \alpha \leqslant 1$. But when these 'higher modes' were used as initial guesses in the present numerical scheme, they failed to produce converged results. Thus they must be regarded as only eigenvalues of the finite but not the semi-infinite problem.

4.2. *The discrete mode for* $-0.16054 \lesssim s \lesssim 0.07452$

Next we consider the steady similarity solutions in the neighbourhood of the Kármán problem for which multiple families of solutions have recently been found by Zandbergen & Dijkstra (1977) and Lentini & Keller (1980). In this case it is convenient to let Ω_0 be the reference angular speed of the basic flow and $s = \Omega_\infty/\Omega_0$. With $\gamma = s$ the time-independent form of (2.3) together with the boundary conditions (2.4) are solved by the techniques described by Zandbergen & Dijkstra (1977). The

Branch I			
s	λ_1	s	λ_1
−0.001	−0.42860	−0.145	−0.25292
−0.020	−0.43479	−0.150	−0.22221
−0.040	−0.43406	−0.155	−0.17777
−0.060	−0.42565	−0.156	−0.16544
−0.080	−0.40896	−0.157	−0.15088
−0.100	−0.38235	−0.158	−0.13281
−0.120	−0.34202	−0.159	−0.10838
−0.130	−0.31397	−0.160	−0.06789
−0.140	−0.27697	−0.1605	−0.01865

Branch II			
s	λ_1	s	λ_1
−0.160125	0.05811	−0.110	0.37286
−0.160	0.06574	−0.100	0.39068
−0.159	0.10547	−0.080	0.41753
−0.155	0.17855	−0.060	0.43601
−0.150	0.22681	−0.050	0.44281
−0.145	0.25971	0.000	0.45745
−0.140	0.28496	0.020	0.45518
−0.135	0.30551	0.040	0.44821
−0.130	0.32283	0.060	0.43601
−0.125	0.33777	0.070	0.42766
−0.120	0.35087	0.072	0.42578
−0.115	0.36248	0.074525625	0.4233088

Branch III			
s	λ_1	s	λ_1
0.0739	0.42393	0.060	0.43602
0.070	0.42766	0.040	0.44821

TABLE 2. The eigenvalue λ_1 as a function of s along the first three branches of the multiple solutions. (The branch I solution at $s = 0$ corresponds to the classical Kármán solution.)

details of the multiple branches of solutions thus obtained can be found in Zandbergen & Dijkstra (1977) and Lentini & Keller (1980) and need not be repeated here; we shall use the same nomenclature as in these earlier papers and refer to the first three branches of the multiple solutions as branches I, II and III. In contrast to the results of §4.1, however, we find only a single real eigenvalue λ_1 for the values of s considered. Our results are presented in figure 5 and they are tabulated in table 2.

Consider first the solutions belonging to branch I for which stability calculations were made for $-0.16054 < s < 0$. We found that the value of λ_1 decreases from 0 at $s = -0.16054$ and it remains negative in the interval, indicating that the steady-state solutions are stable to small perturbations.

On the other hand, it can be seen from figure 5 that $\lambda_1 > 0$ for all the branch II solutions. Although we have not extended our stability calculations beyond the beginning of the third branch, the overall trend of the results in figure 5 strongly suggests that all the recently discovered multiple solutions are unstable to small perturbations.

We also wish to note that we have not been able to locate any discrete modes associated with the branch I solutions for $s \geqslant 0$. The spectral properties of the present stability problem, especially the nature of the discrete spectrum if it exists, is certainly not well understood in the interval $0 \leqslant s < 1$. It will be very useful to clarify this aspect of the problem in the future.

5. Concluding remarks

In this paper we have considered the mathematical stability of the similarity solutions of swirling flow above an infinite rotating disk immersed in an otherwise unbounded rigidly rotating fluid. We have shown that the present stability problem has a continuum spectrum and possibly a finite number of discrete modes. Although the continuum spectrum contains only stable modes, we found that the discrete spectrum can give rise to instability. An important conclusion which can be drawn from our results is that the recently discovered multiple solutions in the neighbourhood of the classical Kármán swirling-flow solution are linearly unstable in a Cauchy sense. Thus, despite the non-uniqueness of these solutions, it is unlikely that any swirling flows other than those belonging to the same family as the classical Kármán solution is physically realizable. A second conclusion that can be drawn is that there exists a critical value $\alpha = \alpha_c \approx -0.03$ that divides the steady similarity solutions in the neighbourhood of the Bödewadt solution into two classes. The steady-state solutions are either stable or unstable depending on whether α is greater or less than α_c. This conclusion is in complete agreement with the conclusion reached by Bodonyi (1978) based on a numerical study of the large-time behaviour of the solutions of the unsteady similarity equations.

We note that the numerical study of Bodonyi (1978) also strongly suggested the existence of certain limit-cycle solutions when the steady-state solutions become unstable for $-0.02 \lesssim \alpha \lesssim \alpha_c$. Moreover, these limit-cycle solutions exhibit behaviour bearing strong resemblances to other supercritical instabilities in hydrodynamics. Thus it might be of some future interest to investigate the effects of the nonlinear self-interaction of the unstable linear mode for $0 < \alpha_c - \alpha \ll 1$ in the same general spirit of the well-known weakly nonlinear hydrodynamic stability theory.

The authors are grateful to Professor O. R. Burggraf for his interest in this work and to the late Professor K. Stewartson for his critical comments on the early drafts of this paper. One of us (B.S.N) also wishes to thank Dr A. Nachman and Professor C. E. Grosch for some useful discussions on the analysis of the continuum spectrum. This work has been supported in part by the Computing Services of Indiana University – Purdue University at Indianapolis and by the National Science Foundation Grants MEA-81-07078 (R. J. B.) and MCS-81-01932 (B.S.N).

Appendix. The numerical procedure

Computation of the eigenvalue

For numerical purposes we choose z_∞ to be sufficiently large such that for $z \geqslant z_\infty$, the coefficient matrix $\boldsymbol{A}$ (cf. (3.4) and (3.7)) is numerically indistinguishable from $\boldsymbol{A}_\infty$. Then on omitting the overall exponential factors in $\boldsymbol{\phi}_2$ and $\boldsymbol{\phi}_3$ the initial conditions for the bounded solutions are given by

$$\left.\begin{aligned} \boldsymbol{\phi}_1 &= [1,0,0,0,0]^{\mathrm{T}}, \\ \boldsymbol{\phi}_2 &= [1/\beta_2, 1, \beta_2, -\mathrm{i}, -\beta_2]^{\mathrm{T}}, \\ \boldsymbol{\phi}_3 &= [1/\beta_3, 1, \beta_3, \mathrm{i}, \mathrm{i}\beta_3]^{\mathrm{T}}. \end{aligned}\right\} \tag{A 1}$$

If the usual shooting method is applied to the present problem, these initial conditions can in principle be used to obtain three linearly independent solutions of (4.2) by integrating from z_∞ to 0. The boundary conditions (4.3*a*) would then require that the eigenvalue λ be chosen (iteratively) such that (cf. (4.4))

$$E_{123} = \det \begin{bmatrix} F_1(0) & F_2(0) & F_3(0) \\ F_1'(0) & F_2'(0) & F_3'(0) \\ G_1(0) & G_2(0) & G_3(0) \end{bmatrix} = 0. \tag{A 2}$$

We note, however, that owing to the length of the interval over which the integration must be performed, numerical instabilities inherent in (4.2) can quickly render $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$ and $\boldsymbol{\phi}_3$ numerically dependent. This in turn can make an accurate determination of λ based on (A 2) difficult.

To overcome these difficulties, we make use of the so-called compound matrix method discussed by Ng & Reid (1979, 1980), which has been shown to be effective in dealing with eigenvalue problems involving unstable equations over a semi-infinite interval. For the present problem, the compound matrix method is based on considering the 3×3 minors of the solution matrix $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1 \vdots \boldsymbol{\phi}_2 \vdots \boldsymbol{\phi}_3]$ rather than attempting to compute $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$ and $\boldsymbol{\phi}_3$ separately.

If we denote the ten minors of $\boldsymbol{\Phi}$ by their respective 3-tuples of row indices in $\boldsymbol{\Phi}$, then, on arranging the 3-tuples in lexicographic order, these minors are given by

$$y_1 = (1,2,3), \quad y_2 = (1,2,4), \quad \ldots, \quad y_{10} = (3,4,5). \tag{A 3}$$

In particular, we note that the eigenvalue relation (A 2) is equivalent to requiring that the minor y_2 vanish at $z = 0$, i.e.

$$y_2(0) = 0. \tag{A 4}$$

If we now let $\boldsymbol{y} = [y_1, y_2, \ldots, y_{10}]^{\mathrm{T}}$, then $\boldsymbol{y}$ is called the third compound of $\boldsymbol{\Phi}$. By using the results of Schwarz (1970) or by differentiating (A 3) and using (4.2), it is easy to show that $\boldsymbol{y}$ satisfies the equation

$$\boldsymbol{y}' = \boldsymbol{B}\boldsymbol{y}, \tag{A 5}$$

where

$$\boldsymbol{B} = \begin{bmatrix}
-2f_0 & -2g_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda+2f_0' & -2f_0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda+2f_0' & 0 & -2f_0 & 1 & 0 & 1 & 0 & 0 & 0 \\
-2g_0 & 0 & \lambda+2f_0' & \lambda+2f_0' & -4f_0 & -2g_0 & 0 & 1 & 0 & 0 \\
0 & -2g_0 & 0 & 0 & 0 & -2f_0 & 0 & 0 & 1 & 0 \\
0 & 2f_0'' & 0 & 0 & 0 & 0 & -2f_0 & 1 & 0 & 0 \\
-2g_0' & 0 & 2f_0'' & 0 & 0 & 0 & \lambda+2f_0' & -4f_0 & -2g_0 & 0 \\
0 & -2g_0' & 0 & 0 & 0 & 0 & 0 & 0 & -2f_0 & 1 \\
0 & 0 & 0 & -2g_0' & 0 & -2f_0'' & 2g_0 & 0 & \lambda+2f_0' & -4f_0
\end{bmatrix} \tag{A 6}$$

On substituting (A 1) into (A 3), we obtain

$$\boldsymbol{y}(z_\infty) = [\beta_3-\beta_2, 2\mathrm{i}, \mathrm{i}(\beta_3+\beta_2), \mathrm{i}(\beta_3+\beta_2), 2\mathrm{i}\beta_3\beta_2, \beta_3-\beta_2, 0, 0, 0, 0]^{\mathrm{T}}. \tag{A 7}$$

The application of the compound matrix method to the present problem thus involves the repeated integration of (A 5) from z_∞ to 0 subject to the initial conditions (A 7). Simultaneously, an iterative procedure such as Newton's method must be used to iterate on λ until the eigenvalue relation (A 2) is satisfied.

Computation of the eigenfunction

Once the required eigenvalue has been obtained by the procedure just described, we can proceed to the determination of the corresponding eigenfunction $\boldsymbol{\phi}$. First we write

$$\boldsymbol{\phi} = c_1 \boldsymbol{\phi}_1 + c_2 \boldsymbol{\phi}_2 + c_3 \boldsymbol{\phi}_3. \tag{A 8}$$

In the present scheme $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$ and $\boldsymbol{\phi}_3$ are not known explicitly. However, on rewriting (A 8) in component form and on eliminating c_1, c_2 and c_3 systematically in five different ways, we obtain

$$y_1 G - y_2 F''' + y_4 F' - y_7 F = 0, \tag{A 9}$$

$$y_1 G' - y_3 F''' + y_5 F' - y_8 F = 0, \tag{A 10}$$

$$y_2 G' - y_3 G + y_6 F' - y_9 F = 0, \tag{A 11}$$

$$y_4 G' - y_5 G + y_6 F''' - y_{10} F = 0, \tag{A 12}$$

$$y_7 G' - y_8 G + y_9 F''' - y_{10} F' = 0. \tag{A 13}$$

For a detailed discussion of the general technique and the rationale used in the derivation of (A 9)–(A 13), readers may refer to Ng & Reid (1984). For our purpose, it is sufficient to note that (A 9) and (A 11) form a closed system of the form

$$\begin{bmatrix} F \\ F' \\ G \end{bmatrix}' = \frac{1}{y_2} \begin{bmatrix} 0 & y_2 & 0 \\ -y_7 & y_4 & y_1 \\ y_9 & -y_6 & y_3 \end{bmatrix} \begin{bmatrix} F \\ F' \\ G \end{bmatrix}. \tag{A 14}$$

This then suggests that F, F' and G can be obtained by integrating (A 14) subject to some appropriate initial conditions. Once F, F' and G are determined, G' and F''' can be computed algebraically using, for example, (A 11) and then (A 12) or (A 13). It is clear, of course, that (A 14) is singular at $z = 0$ and hence it is not possible to start the integration from the origin. Nevertheless, if we fix the normalization of $\boldsymbol{\phi}$ by letting $F'''(0) = 1$ then on substituting (4.3*a*) into (A 10) we have $G'(0) = y_3(0)/y_1(0)$. Thus the initial conditions for $\boldsymbol{\phi}$ at $z = 0$ are completely specified, and (4.2) can be integrated one step forward from 0 to h (say). The value of $\boldsymbol{\phi}(h)$ now provides the necessary initial condition for (A 14).

Next we note that as $z \to \infty$ (A 14) has the asymptotic form

$$\begin{bmatrix} F \\ F' \\ G \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2}(\beta_3 + \beta_2) & -\frac{1}{2}\mathrm{i}(\beta_3 - \beta_2) \\ 0 & \frac{1}{2}\mathrm{i}(\beta_3 - \beta_2) & \frac{1}{2}(\beta_3 + \beta_2) \end{bmatrix} \begin{bmatrix} F \\ F' \\ G \end{bmatrix}. \tag{A 15}$$

A simple calculation shows that the roots of the characteristic equation associated with (A 15) are 0, β_2 and β_3. Thus as $z \to \infty$ any solution of (A 14) is necessarily a linear combination of $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$ and $\boldsymbol{\phi}_3$, and the boundary conditions (4.3*b*) are therefore automatically satisfied. On the other hand, if one were to compute the eigenfunction $\boldsymbol{\phi}$ by integrating (4.2) directly, the resulting solution will quickly become numerically unstable for large z owing to the inevitable presence of some multiples of the unbounded solutions $\boldsymbol{\phi}_4$ and $\boldsymbol{\phi}_5$.

REFERENCES

BODONYI, R. J. 1973 Ph.D. thesis, The Ohio State University.

BODONYI, R. J. 1975 *J. Fluid Mech.* **67**, 657.

BODONYI, R. J. 1978 *Q. J. Mech. Appl. Maths* **31**, 461.

BODONYI, R. J. & STEWARTSON, K. 1977 *J. Fluid Mech.* **79**, 699.

EVANS, D. J. 1969 *Q. J. Mech. Appl. Maths* **22**, 467.

GUSTAVSSON, L. H. 1979 *Phys. Fluids* **22**, 1602.

GROSCH, C. E. & SALWEN, H. 1978 *J. Fluid Mech.* **87**, 33.

KÁRMÁN, T. VON 1921 *Z. angew. Math. Mech.* **1**, 223.

LENTINI, M. & KELLER, H. B. 1980 *SIAM J. Appl. Maths* **38**, 52.

MCLEOD, J. B. 1970 *Mathematika* **17**, 243.

MURDOCK, J. N. & STEWARTSON, K. 1977 *Phys. Fluids* **20**, 1404.

NG, B. S. & REID, W. H. 1979 *J. Comp. Phys.* **30**, 125.

NG, B. S. & REID, W. H. 1980 *J. Comp. Phys.* **38**, 275.

NG, B. S. & REID, W. H. 1984 *J. Comp. Phys.* (in press).

SCHWARZ, B. 1970 *Pac. J. Maths* **32**, 203.

STEWARTSON, K., SIMPSON, C. J. & BODONYI, R. J. 1982 *J. Fluid Mech.* **121**, 507.

ZANDBERGEN, P. J. & DIJKSTRA, D. 1977 *J. Engng Maths* **11**, 669.